

Director: Arthur Erdelyi

Research Associates: Wilhelm Magnus (1948-50)

Fritz Oberhettinger (1948-51) Francesco G. Tricomi (1948-51)

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CHAPTER XVIII

MISCELLANEOUS FUNCTIONS

18.1. Mittag-Leffler's function $E_\alpha(z)$ and related functions

The function

$$(1) \quad E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

was introduced by Mittag-Leffler (1903, 1904, 1905) and was investigated by several authors among whom we mention Wiman (1905), Pollard (1948), Humbert (1953). In this chapter E will always stand for the function (1) which must not be confused with the physicists' notation for the incomplete gamma function mentioned in sec. 9.2.

$E_\alpha(z)$, for $\alpha > 0$, furnishes important examples of entire functions of any given finite order: in a certain sense each $E_\alpha(z)$ is the simplest entire function of its order (Phragmén 1904). Mittag-Leffler's function also furnishes examples and counter-examples for the growth and other properties of entire functions of finite order, and has other applications (Buhl 1925).

We have

$$(2) \quad E_1(z) = e^z, \quad E_2(z^2) = \cosh z, \quad E_{1/2}(z^{1/2}) = 2\pi^{-1/2} e^{-z} \operatorname{Erfc}(-z^{1/2})$$

and $E_n(z^n)$ for positive integer n is a generalized hyperbolic function (see also sec. 18.2).

Many of the most important properties of $E_\alpha(z)$ follow from Mittag-Leffler's integral representation

$$(3) \quad E_\alpha(z) = \frac{1}{2\pi i} \int_C \frac{t^{\alpha-1} e^t}{t^\alpha - z} dt$$

where the path of integration C is a loop which starts and ends at $-\infty$,

and encircles the circular disc $|t| \leq |z|^{1/a}$ in the positive sense: $-\pi \leq \arg t \leq \pi$ on C . To prove (3), expand the integrand in powers of z , integrate term-by-term, and use Hankel's integral 1.6(2) for the reciprocal of the gamma function.

The integrand in (3) has a branch-point at $t = 0$. The complex t -plane is cut along the negative real axis, and in the cut plane the integrand is single-valued: the principal branch of t^a is taken in the cut plane. The integrand has poles at the points,

$$(4) \quad t_m = z^{1/a} e^{2\pi i m/a} \quad m \text{ integer}$$

but only those of the poles lie in the cut plane for which

$$(5) \quad -a\pi < \arg z + 2\pi m < a\pi.$$

Thus, the number of the poles inside C is either $[a]$ or $[a + 1]$, according to the value of $\arg z$.

Feller conjectured and Pollard (1948) proved that $E_a(-x)$ is *completely monotonic* for $x \geq 0$ if $0 \leq a \leq 1$, i.e., that

$$(6) \quad (-1)^n \frac{d^n E_a(-x)}{dx^n} \geq 0 \quad x \geq 0, \quad 0 \leq a \leq 1$$

The proof is based on (3).

To investigate the asymptotic behavior of $E_a(z)$ as $z \rightarrow \infty$, first assume that $z \rightarrow \infty$ along a ray which is outside the sector $|\arg z| \leq a\pi/2$ (there are such rays if $0 < a < 2$). If there are any poles t_m satisfying (5), they will lie in the half-plane $\operatorname{Re} t < 0$. Deform C to consist of two rays in the half-plane $\operatorname{Re} t < 0$ so that the poles, if any, lie to the left of C , also set

$$\frac{t^a}{t^a - z} = - \sum_{n=1}^{N-1} \frac{t^{na}}{z^n} - \left(1 - \frac{t^a}{z}\right)^{-1} \frac{t^{Na}}{z^N}$$

in (3) and note that $(1 - t^a z^{-1})^{-1}$ is bounded uniformly in $|z|$ and t if $\arg z$ is constant and t is on C . Using again 1.6(2), the result is

$$(7) \quad E_a(z) = - \sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(1 - an)} + O(|z|^{-N})$$

$z \rightarrow \infty, \quad |\arg(-z)| < (1 - \frac{1}{2}a)\pi$

The O -term is uniform in $\arg z$ if

$$|\arg(-z)| \leq (1 - \frac{1}{2}a - \epsilon)\pi, \quad \epsilon > 0$$

The result is vacuous when $a \geq 2$.

Next assume that $z \rightarrow \infty$ along a ray, and $|\arg z| \leq a\pi/2$. Then there is at least one t_n satisfying

$$(8) \quad -\frac{1}{2}a\pi \leq \arg z + 2\pi m \leq \frac{1}{2}a\pi,$$

and there may be several (if $a \geq 2$): these poles lie in the half-plane $\operatorname{Re} t \geq 0$. C can now be deformed as before except that in the course of the deformation of C the poles satisfying (8) are crossed and contribute residues. The result then is

$$(9) \quad E_a(z) = \frac{1}{a} \sum_n e^{t_n} - \sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(1-an)} + O(|z|^{-N})$$

$$z \rightarrow \infty, \quad |\arg z| \leq \frac{1}{2}a\pi$$

where t_n is given by (4) and summation is over all those integers m which satisfy (8). In particular, if $0 < a < 2$, $m = 0$ is the only integer satisfying (8), and

$$(10) \quad E_a(z) = \frac{1}{a} \exp z^{1/a} + O(|z|^{-1}) \quad 0 < a < 2, \quad |\arg z| \leq \frac{1}{2}a\pi, \quad z \rightarrow \infty$$

From (7), (9), (10), and the definition of the order of an entire function (see, for instance, Copson 1935, sec. 7.4) we infer that $E_a(z)$ is an *entire function of order* $1/a$ for $a > 0$. The asymptotic expansions (7), (9) were generalized to complex values of a by Wiman (1905).

The zeros of $E_a(z)$ were investigated by Wiman (1905). For $a \geq 2$ Wiman proved that $E_a(z)$ has an infinity of zeros on the negative real axis, and it has no other zeros. If $n(r)$ is the number of zeros of $E_a(z)$ in $|z| < r$, Wiman proved

$$(11) \quad \left[\frac{r^{1/a}}{\pi} \sin \frac{\pi}{a} \right] \leq n(r) < \left[\frac{r^{1/a}}{\pi} \sin \frac{\pi}{a} \right] + 1 \quad a \geq 2$$

where $[x]$ is the greatest integer $\leq x$. For $0 < a < 2$ the distribution of zeros is entirely different. Excluding the case $a = 1$ (when there are no zeros), Wiman shows that asymptotically the zeros lie on the curve

$$(12) \quad \operatorname{Re} z^{1/a} + \log |z| + \log |\Gamma(-a)| = 0$$

and also that

$$(13) \quad [\pi^{-1} r^{1/\alpha} - \frac{1}{2}\alpha] - 1 \leq n(r) \leq [\pi^{-1} r^{1/\alpha} - \frac{1}{2}\alpha] + 1 \quad 0 < \alpha < 2, \quad \alpha \neq 1$$

Moreover, for $1 < \alpha < 2$, there is an odd number of negative zeros. Wiman investigated the zeros of $E_\alpha(z)$ also for complex values of α .

The functional relations

$$(14) \quad \sum_{h=0}^{n-1} E_\alpha(z e^{2\pi i h/n}) = m E_{n\alpha}(z^n)$$

$$(15) \quad \left(\frac{d}{dz}\right)^n E_n(z^n) = E_n(z^n)$$

$$(16) \quad \left(\frac{d}{dz}\right)^m E_{n/n}(z^{n/n}) = \sum_{k=1}^{n-1} \frac{z^{-km/n}}{\Gamma(1 - km/n)} + E_{n/n}(z^{n/n})$$

where m and $n - 1$ are positive integers, are immediate consequences of (1). From (16)

$$\frac{d}{dz} [e^{-z} E_{1/n}(z^{1/n})] = e^{-z} \sum_{k=1}^{n-1} \frac{z^{-k/n}}{\Gamma(1 - k/n)}$$

and upon integration of this by means of 9.1(1)

$$(17) \quad E_{1/n}(z^{1/n}) = e^z \left[1 + \sum_{k=1}^{n-1} \frac{\gamma(1 - k/n, z)}{\Gamma(1 - k/n)} \right] \quad n = 2, 3, \dots$$

An explicit expression for $E_{n/n}$ follows from (14) and (17). The third equation (2) follows from (17) for $n = 2$ by means of 9.9(1), (2).

The integral

$$(18) \quad \int_0^\infty e^{-t} E_\alpha(t^\alpha z) dt = \frac{1}{1-z} \quad \alpha \geq 0$$

was evaluated by Mittag-Leffler who showed that the region of convergence of (18) contains the unit circle and is bounded by the line $\operatorname{Re} z^{1/\alpha} = 1$. The Laplace transform of $E_\alpha(t^\alpha)$ may be obtained from (18), and was used by Humbert (1953) to obtain a number of functional relations satisfied by $E_\alpha(z)$.

integer. The region of convergence of (26) is the same as that of (18). The Laplace transform of $t^{\beta-1} E_{\alpha}(t^{\alpha})$ may be evaluated by means of (26) and was used by Agarwal (1953) and by Humbert and Agarwal (1953) to obtain further properties of $E_{\alpha, \beta}$.

A function of two variables resembling $E_{\alpha, \beta}$ was briefly discussed by Humbert and Delerue (1953).

The functions E_{α} and $E_{\alpha, \beta}$ increase indefinitely as $z \rightarrow \infty$ in a certain sector of angle $\alpha\pi$, and approach zero as $z \rightarrow \infty$ outside of this sector. Entire functions which increase indefinitely in a single direction, and approach zero in all other directions, are also known. Two such functions are

$$\sum_{k=2}^{\infty} \frac{z^k}{\Gamma[1+k(\log k)^{-\alpha}]} \quad 0 < \alpha < 1$$

$$\sum_{k=0}^{\infty} \left[\frac{z}{\log(k+1/\alpha)} \right]^k \quad 0 < \alpha < 1$$

They have been discussed, respectively, by Malmquist (1905) and Lindelöf (1903).

Barnes (1906) has investigated the asymptotic behavior of $E_{\alpha}(z)$, and also that of several similar functions, in particular of the functions

$$\sum_{k=0}^{\infty} \frac{z^k}{(k+\theta)^{\beta} \Gamma(1+ak)}, \quad \sum_{k=0}^{\infty} \frac{z^k \Gamma(1+ak)}{k!},$$

$$\sum_{k=0}^{\infty} \frac{z^k \Gamma(1+ak)}{\Gamma(1+a+ak)}.$$

A function intimately connected with $E_{\alpha, \beta}$ is the entire function

$$(27) \quad \phi(\alpha, \beta; z) = \sum_{k=0}^{\infty} \frac{z^k}{k! \Gamma(ak + \beta)} \quad \alpha, \beta > 0$$

which was used by Wright (1934) in the asymptotic theory of partitions. The connection with $E_{\alpha, \beta}$ is given by

$$(28) \quad \int_0^{\infty} e^{-ts} \phi(\alpha, \beta; t) dt = s^{-1} E_{\alpha, \beta}(s^{-1}) \quad \alpha > 1, \quad \beta > 0$$

$\phi(z)$ can be represented by the integral (Wright 1933)

$$(29) \quad \phi(a, \beta; z) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} u^{-\beta} \exp(u + zu^{-a}) du \quad a > 0$$

To prove (29), expand the integrand in powers of z and use 1.6(2). The asymptotic behavior of ϕ as $z \rightarrow \infty$ was also investigated by Wright (1934 a, 1940). The relations

$$(30) \quad az \phi(a, a + \beta; z) = \phi(a, \beta - 1; z) + (1 - \beta) \phi(a, \beta; z)$$

$$(31) \quad \frac{d\phi(a, \beta; z)}{dz} = \phi(a, a + \beta; z)$$

$$(32) \quad az \frac{d\phi(a, a + \beta; z)}{dz} = \phi(a, \beta - 1; z) + (1 - \beta) \phi(a, \beta; z)$$

follow from (27). Since

$$(33) \quad J_\nu(z) = (\frac{1}{2}z)^\nu \phi\left(1, \nu + 1; -\frac{z^2}{4}\right),$$

Wright's function may be regarded as a kind of generalized Bessel function. (30) is a generalization of the recurrence relation of Bessel functions, and (31), (32) are generalizations of the differentiation formulas. Some of the properties which ϕ shares with Bessel functions were enumerated by Wright. A generalized Hankel transformation with the kernel

$$(\frac{1}{2})^{\beta-1} (xy)^{\beta-\frac{1}{2}} \phi\left(a, \beta; -\frac{x^2 y^2}{4}\right)$$

was discussed by Agarwal (1950, 1951, 1953 a).

18.2. Trigonometric and hyperbolic functions of order n

In this section n will be a positive integer and

$$(1) \quad \omega = \exp\left(\frac{2\pi i}{n}\right)$$

The n functions

$$(2) \quad h_i(x, n) = \frac{1}{n} \sum_{m=1}^n \omega^{(1-i)m} \exp(\omega^m x) \quad i = 1, 2, \dots, n$$

are sometimes called *hyperbolic functions of order n*. They reduce to hyperbolic functions when $n = 2$.

$$(3) \quad h_1(x, 1) = e^x, \quad h_1(x, 2) = \cosh x, \quad h_2(x, 2) = \sinh x$$

In general, n will be a fixed positive integer and will, as a rule, not be indicated. It will also be convenient to extend the definition (2) to all (positive, zero, or negative) integers i which is tantamount to setting

$$(4) \quad h_{i+n}(x, n) = h_i(x, n) \qquad i \text{ integer}$$

This will often simplify the writing of formulas.

Since $\omega^n = 1$, all h_i satisfy the differential equation

$$(5) \quad \frac{d^n y}{dx^n} - y = 0$$

and since

$$(6) \quad \sum_{r=1}^n \omega^{rn} = 0 \quad \text{for integers } r \text{ not divisible by } n \\ = n \quad \text{for integers } r \text{ divisible by } n,$$

the h_i also satisfy the initial conditions

$$(7) \quad \frac{d^{j-1} h_i}{dx^{j-1}}(0) = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \\ i, j = 1, 2, \dots, n$$

Thus, h_1, \dots, h_n form a linearly independent set of solutions of (5), and their Wronskian is equal to unity.

The power series expansion

$$(8) \quad h_i(x, n) = \sum_{r=0}^{\infty} \frac{x^{nr+i-1}}{(nr+i-1)!} \qquad i = 1, 2, \dots, n$$

is obtained by expanding the exponential functions in (2) and using (6); the integral representation

$$(9) \quad h_i(x, n) = \frac{1}{2\pi i} \int_C \frac{t^{n-i} e^{xt}}{t^n - 1} dt \qquad i = 1, \dots, n$$

where C is a simple closed curve encircling the unit circle once in the positive sense, is proved by the remark that the evaluation of the integral as a sum of residues leads to (2); and the relation

$$(10) \exp(\omega^m x) = \sum_{i=1}^n \omega^{(i-1)m} h_i(x, n) \quad m \text{ integer}$$

follows from (8).

Some of the basic formulas for hyperbolic functions of order n are

$$(11) h_i(\omega^m x) = \omega^{(i-1)m} h_i(x)$$

$$(12) \frac{d^j h_i(x)}{dx^j} = h_{i-j}(x)$$

$$(13) h_i(x + y) = \sum_{j=1}^n h_j(x) h_{i-j+1}(y)$$

$$(14) \begin{vmatrix} h_1 & h_2 & \dots & h_n \\ h_n & h_1 & \dots & h_{n-1} \\ \dots & \dots & \dots & \dots \\ h_2 & h_3 & \dots & h_1 \end{vmatrix} = \prod_{m=1}^n \left(\sum_{i=1}^n \omega^{(i-1)m} h_i(x) \right) = 1$$

$$(15) \int_0^\infty e^{-st} h_i(t) dt = \frac{s^{n-i}}{s^n - 1} \quad \text{Re } s > 1, \quad i = 1, 2, \dots, n$$

Here i, j, m are any integers [except in (15) where i is restricted]. (11) and (12) follow from (2), (13) follows from (5) since $h_i(x + a)$ is that solution of the differential equation (5) whose j -th derivative is $h_{i-j}(a)$ when $x = 0$, (14) is the Wronskian of h_1, \dots, h_n which is a *circulant* (see Aitken 1939, sec. 51) and can be evaluated explicitly, and (15) is the Laplace transform of $h_i(t)$ and follows likewise from (2) or (8).

For these and other formulas see Poli (1940, 1949a, the latter with a detailed bibliography), Oniga (1948), Bruwier (1949, 1949a), and Silverman (1953). Poli (1949a) indicates some relations which hold when n is a composite number, gives expansions in terms of the h_i , and some applications. Bruwier (1949b) considers $1, \omega, \omega^2, \dots, \omega^{n-1}$ as the units of a linear algebra, the multiplication table being specified by

$$\omega^i \cdot \omega^j = \omega^{i+j}$$

(hypercomplex numbers). $e^{\omega x}$ is a hypercomplex number, and (10) shows that the h_i are the components of $e^{\omega x}$. This fact is used by Bruwier to prove the properties of the $h_i(x)$. Matrices whose elements in the i -th row and j -th column are $a_i h_{j-i}(x, n)/a_j$, where $i, j = 1, 2, \dots, n$ and a_1, \dots, a_n is a given set of constants, were considered by Lehrer (1954).

From (8) and 18.1 (19),

$$(16) \quad h_i(x) = x^{i-1} E_{n,i}(x^n) \quad i = 1, 2, \dots, n$$

and in particular

$$(17) \quad h_1(x) = E_n(x^n)$$

giving the connection with Mittag-Leffler's function.

The n functions

$$(18) \quad k_i(x, n) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{nr+i-1}}{(nr+i-1)!} \quad i = 1, 2, \dots, n$$

are sometimes called *trigonometric functions of order n* ; they are solutions of the differential equation

$$(19) \quad \frac{d^n y}{dx^n} + y = 0$$

and satisfy the initial conditions

$$(20) \quad \frac{d^{j-1} k_i}{dx^{j-1}}(0) = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \quad i, j = 1, 2, \dots, n$$

Here again we extend the definition to all integers i by setting

$$(21) \quad k_{i+n}(x, n) = -k_i(x, n).$$

These functions have been investigated by the above-mentioned authors and also by Mikusiński (1948). With

$$(22) \quad \lambda = \exp\left(\frac{\pi i}{n}\right)$$

so that λ is an n -th root of -1 , we have

$$(23) \quad k_i(x) = \lambda^{1-i} h_i(\lambda x, n)$$

and the properties of the k_i follow easily from those of the h_i . The principal formulas are

$$(24) \quad k_i(\lambda x) = \lambda^{i-1} h_i(x)$$

$$(25) \quad k_i(\omega^n x) = \omega^{(i-1)n} k_i(x)$$

$$(26) \quad \frac{d^j k_i(x)}{dx^j} = k_{i-j}(x)$$

$$(27) \quad k_i(x) = \frac{1}{n} \sum_{m=1}^n \lambda^{(1-i)(2m+1)} \exp(\lambda^{2m+1} x)$$

$$(28) \quad \exp(\lambda^{2m+1} x) = \sum_{i=1}^n \lambda^{(i-1)(2m+1)} k_i(x)$$

$$(29) \quad k_i(x) = \frac{1}{2\pi i} \int_C \frac{t^{n-i} e^{xt}}{t^n + 1} dt$$

$$(30) \quad k_i(x+y) = \sum_{j=1}^n k_j(x) k_{i-j+1}(y)$$

$$(31) \quad \prod_{m=1}^n \left(\sum_{i=1}^n \lambda^{(i-1)(2m+1)} k_i(x) \right) = 1$$

$$(32) \quad \int_0^\infty e^{-st} k_i(t) dt = \frac{s^{n-i}}{s^n + 1} \quad \text{Re } s > 1, \quad i = 1, \dots, n$$

$$(33) \quad h_i(x, n) + k_i(x, n) = 2h_i(x, 2n)$$

$$h_i(x, n) - k_i(x, n) = 2h_{n+i}(x, 2n)$$

It can be seen from (27) that $k_i(x, n)$ is not a periodic function except for $n = 1, 2$. The zeros of $k_i(x)$ have been investigated by Poli (1949 a) for $n = 3$ and by Mikusiński (1948) for any $n > 1$. Mikusiński's investigations are based on the system of linear differential equations satisfied

by $k_1(x), \dots, k_n(x)$ and lead to the following conclusions. Each $k_i(x, n)$ has an infinity of simple positive zeros: the zeros of $k_i(x, n)$ and $k_j(x, n)$, $i \not\equiv j \pmod{n}$ interlace. The least positive zero of $k_i(x, n)$ is between

$$\left[\frac{(i+n-1)!}{(i-1)!} \right]^{1/n} \quad \text{and} \quad \left[\frac{2(i+n-1)!}{(i-1)!} \right]^{1/n}$$

The large positive zeros of $k_i(x, n)$ are approximately equally spaced, the distance between two consecutive zeros of $k_i(x, n)$ approaches $\pi \csc(\pi/n)$.

Quotients like $k_i(x, n)/k_j(x, n)$ may be regarded as generalizations of $\tan x$ and $\operatorname{ctn} x$: for these generalizations see Oniga (1948), Poli (1949).

An entirely different generalization of trigonometric functions was given by Grammel (1948, 1948a, 1950).

18.3. The function $\nu(x)$ and related functions

The functions to be considered in this section are

$$(1) \quad \nu(x) = \int_0^\infty \frac{x^t dt}{\Gamma(t+1)}, \quad \nu(x, a) = \int_0^\infty \frac{x^{a+t} dt}{\Gamma(a+t+1)}$$

$$(2) \quad \mu(x, \beta) = \int_0^\infty \frac{x^t t^\beta dt}{\Gamma(\beta+1) \Gamma(t+1)}$$

$$\mu(x, \beta, a) = \int_0^\infty \frac{x^{a+t} t^\beta dt}{\Gamma(\beta+1) \Gamma(a+t+1)}$$

The first of these functions was encountered by Volterra in his theory of convolution-logarithms (Volterra 1916, Chapter VI, Volterra and Pères 1924, Chapter X): Volterra denoted $\nu(y-x)$ by $\lambda(x, y)$, and $\nu(y-x, a)$ by $\lambda(x, y; a)$ or $\lambda(x, y|a)$. These functions also occur in connection with operational calculus, appear in an inversion formula of the Laplace transformation, and are of interest in connection with certain integral equations. It may be noted that (2) is the definition of μ adopted in recent papers; some of the older papers write μ for a function which differs from (2) by a factor $\Gamma(\beta+1)$.

Between the four functions defined by (1), (2) we have the following relations

$$(3) \quad \nu(x) = \nu(x, 0) = \mu(x, 0) = \mu(x, 0, 0)$$

$$\nu(x, \alpha) = \mu(x, 0, \alpha), \quad \mu(x, \beta) = \mu(x, \beta, 0) = x \mu(x, \beta - 1, -1)$$

$$x \nu(x, \alpha - 1) - \alpha \nu(x, \alpha) = \mu(x, 1, \alpha)$$

All integrals in (1), (2) converge if $x \neq 0$, α is arbitrary, and $\text{Re } \beta > -1$. All four functions are analytic functions of x with branch-points at $x = 0$ and ∞ , and no other singularity; $\nu(x, \alpha)$ and $\mu(x, \beta, \alpha)$ are entire functions of α . The definition of μ can be extended to the entire β -plane by repeated integrations by parts. From (2) it follows that

$$(4) \quad \mu(x, \beta, \alpha) = \int_0^\infty \frac{x^{\alpha+t}}{\Gamma(\alpha+t+1)} d \left[\frac{t^{\beta+1}}{\Gamma(\beta+2)} \right]$$

$$= -\frac{1}{\Gamma(\beta+2)} \int_0^\infty t^{\beta+1} \frac{d}{dt} \left[\frac{x^{\alpha+t}}{\Gamma(\alpha+t+1)} \right] dt$$

$$= \dots \dots \dots$$

$$= \frac{(-1)^m}{\Gamma(\beta+m+1)} \int_0^\infty t^{\beta+m} \frac{d^m}{dt^m} \left[\frac{x^{\alpha+t}}{\Gamma(\alpha+t+1)} \right] dt$$

and the last expression may be regarded as the definition of $\mu(x, \beta, \alpha)$ for $\text{Re } \beta > -m - 1$. The so extended functions $\mu(x, \beta, \alpha)$ and $\mu(x, \beta) = \mu(x, \beta, 0)$ are entire functions of β , and they are analytic functions of x , and $\mu(x, \beta, \alpha)$ is also an entire function of α .

From (4) it follows that

$$(5) \quad \mu(x, -m, \alpha) = (-1)^{m-1} \frac{d^{m-1}}{d\alpha^{m-1}} \left[\frac{x^\alpha}{\Gamma(\alpha+1)} \right] \quad m = 1, 2, \dots$$

and since $x^\alpha/\Gamma(\alpha+1)$ is an entire function of α , we have by Taylor's expansion

$$(6) \quad \frac{x^{\alpha+t}}{\Gamma(\alpha+t+1)} = \sum_{n=0}^\infty \mu(x, -n-1, \alpha) \frac{(-t)^n}{n!}$$

In order to investigate the behavior of $\mu(x, \beta, \alpha)$ as $x \rightarrow 0$, we rewrite the second formula (2) as

$$(7) \quad \Gamma(\beta + 1) \mu(x, \beta, \alpha) = x^\alpha \int_0^\infty \exp\left(-t \log \frac{1}{x}\right) \frac{t^\beta dt}{\Gamma(\alpha + t + 1)}$$

From (6) we have

$$(8) \quad \frac{1}{\Gamma(\alpha + t + 1)} = \sum_{n=0}^{\infty} \mu(1, -n - 1, \alpha) \frac{(-t)^n}{n!}$$

and it is known from Watson's lemma (Copson 1935, sec. 9.52) that substitution of (8) in (7) and integration term-by-term will give the asymptotic expansion of the integral in descending powers of $\log(1/x)$. Thus,

$$(9) \quad \mu(x, \beta, \alpha) = x^\alpha \left(\log \frac{1}{x}\right)^{-\beta-1} \left[\sum_{n=0}^{N-1} \frac{(-1)^n (\beta + 1)_n}{n!} \right. \\ \left. \times \mu(1, -n - 1, \alpha) \left(\log \frac{1}{x}\right)^{-n} + O\left(\left|\log \frac{1}{x}\right|^{-N}\right) \right] \\ \text{Re } \beta > -1, \quad x \rightarrow 0, \quad \left| \arg\left(\log \frac{1}{x}\right) \right| < \pi$$

The asymptotic expansions of the other three functions in descending powers of $\log(1/x)$ follow by (3). The first terms of the asymptotic expansions of $\nu(x)$ and of $\nu(x, \alpha)$ were obtained by Volterra.

The behavior of $\nu(x)$ as $\text{Re } x \rightarrow \infty$ can be seen from Ramanujan's integral (Hardy 1940, p. 196)

$$(10) \quad \nu(x) = e^x - \int_0^\infty \frac{e^{-xt} dt}{t[\pi^2 + (\log t)^2]} \quad \text{Re } x > 0$$

A thorough investigation of the asymptotic behavior of $\nu(x)$ was undertaken by Ford (1936). Briefly, Ford's method is as follows. Let us integrate

$$H(x, w) = \frac{1}{[\sin(\pi w)]^2} \int_0^w \frac{x^{\alpha+t} dt}{\Gamma(\alpha + t + 1)}$$

around a rectangle in the w -plane whose corners are $-N - \frac{1}{2} - ic$, $k + \frac{1}{2} - ic$, $k + \frac{1}{2} + ic$, $-N - \frac{1}{2} + ic$ where k and N are integers, $k + N \geq 0$, and c is a positive number. $H(x, w)$ is a meromorphic function, and its poles inside the rectangle are at $w = n$, $n = -N, -N + 1, \dots, k - 1, k$. The residue of H at $w = n$ is $\pi^{-2} x^{\alpha+n} / \Gamma(\alpha + n + 1)$. If $c \rightarrow \infty$, the integrals along the horizontal lines of the rectangle vanish so that

$$\frac{1}{2\pi i} \int_{k+\frac{1}{2}-i\infty}^{k+\frac{1}{2}+i\infty} H dw - \frac{1}{2\pi i} \int_{-N-\frac{1}{2}-i\infty}^{-N-\frac{1}{2}+i\infty} H dw = \sum_{n=-N}^k \frac{\pi^{-2} x^{\alpha+n}}{\Gamma(\alpha+n+1)}$$

Clearly, the second integral is $O(|x|^{\alpha-N-\frac{1}{2}})$. In the first integral we set

$$H(x, w) = H_1 + H_2 = \frac{1}{[\sin(\pi w)]^2} \left(\int_0^{k+\frac{1}{2}} + \int_{k+\frac{1}{2}}^w \right)$$

It can then be shown that

$$\begin{aligned} \frac{1}{2\pi i} \int_{k+\frac{1}{2}-i\infty}^{k+\frac{1}{2}+i\infty} H_1 dw &= \frac{1}{2\pi i} \int_0^{k+\frac{1}{2}} \frac{x^{\alpha+t} dt}{\Gamma(\alpha+t+1)} \int_{k+\frac{1}{2}-i\infty}^{k+\frac{1}{2}+i\infty} [\sin(\pi w)]^{-2} dw \\ &= \pi^{-2} \int_0^{k+\frac{1}{2}} \frac{x^{\alpha+t} dt}{\Gamma(\alpha+t+1)} \end{aligned}$$

$$\frac{1}{2\pi i} \int_{k+\frac{1}{2}-i\infty}^{k+\frac{1}{2}+i\infty} H_2 dw \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty,$$

and hence, making $k \rightarrow \infty$,

$$\begin{aligned} \nu(x, \alpha) - \sum_{n=-N}^{\infty} \frac{x^{\alpha+n}}{\Gamma(\alpha+n+1)} &= -\frac{1}{2} \pi i \int_{-N-\frac{1}{2}-i\infty}^{-N-\frac{1}{2}+i\infty} H dw \\ &= O(|x|^{\alpha-N-\frac{1}{2}}) \quad |x| \rightarrow \infty \end{aligned}$$

Combining this result with 18.1 (21), (22),

$$\begin{aligned} \nu(x, \alpha) &= e^x + O(|x|^{\alpha-N}) & x \rightarrow \infty, \quad |\arg x| \leq \frac{1}{2}\pi \\ &= O(|x|^{\alpha-N}) & x \rightarrow \infty, \quad \frac{1}{2}\pi < |\arg x| \leq \pi \end{aligned}$$

for any integer N .

For $\mu(x, \beta, \alpha)$ a somewhat less complete result can similarly be derived. Because of the branch-point of

$$H(x, w, \beta) = \frac{1}{[\sin(\pi w)]^2} \int_0^w \frac{x^{\alpha+t} t^\beta dt}{\Gamma(\alpha+t+1)}$$

at $w = 0$, one is forced to take $N = -1$ and obtains, as above,

$$\mu(x, \beta, \alpha) - \sum_{n=1}^{\infty} \frac{x^{\alpha+n} n^\beta}{\Gamma(\alpha+n+1)} = -\frac{1}{2}\pi i \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} H(x, w, \beta) dw$$

Further progress then would seem to depend on the asymptotic expansion of the entire function

$$\sum_{n=1}^{\infty} \frac{x^n n^\beta}{\Gamma(\alpha+n+1)}$$

The following recurrence formula, differentiation formulas, series, and integral are easy consequences of (1) and (2).

$$(11) \quad \mu(x, \beta+1, \alpha) = x \mu(x, \beta, \alpha-1) - \alpha \mu(x, \beta, \alpha)$$

$$(12) \quad \frac{d^n \nu(x)}{dx^n} = \nu(x, -n), \quad \frac{d^n \nu(x, \alpha)}{dx^n} = \nu(x, \alpha-n)$$

$$(13) \quad \frac{d^n \mu(x, \beta)}{dx^n} = \mu(x, \beta, -n), \quad \frac{d^n \mu(x, \beta, \alpha)}{dx^n} = \mu(x, \beta, \alpha-n)$$

$$(14) \quad \sum_{n=0}^{\infty} u^n \mu(x, n) = e^{-u} \nu(xe^u), \quad \sum_{n=0}^{\infty} u^n \mu(x, n, \alpha) = e^{-(\alpha+1)u} \nu(xe^u, \alpha)$$

$$\sum_{n=0}^{\infty} \frac{(\beta+1)_n}{n!} u^n \mu(x, \beta+n, \alpha) = e^{-(\alpha+1)u} \mu(xe^u, \beta, \alpha)$$

$$(15) \int_0^{\infty} e^{\alpha u} u^{\gamma-1} \mu(xe^{-u}, \beta, \alpha) du = \frac{\Gamma(\gamma) \Gamma(\beta-\gamma+1)}{\Gamma(\beta+1)} \mu(x, \beta-\gamma, \alpha)$$

$$\operatorname{Re} \beta > -1, \quad \operatorname{Re} \gamma > 0$$

For numerous other formulas regarding these functions see in particular Barrucand (1951), Colombo (1950, 1953), Humbert and Poli (1944).

The occurrence of the functions ν and μ in operational calculus is due on the one hand to the formulas

$$(16) \int_0^{\infty} \frac{e^{-st}}{\Gamma(t+1)} dt = \nu(e^{-s}), \quad \int_0^{\infty} \frac{e^{-st}}{\Gamma(\alpha+t+1)} dt = e^{\alpha s} \nu(e^{-s}, \alpha)$$

$$(17) \int_0^{\infty} \frac{t^{\beta} e^{-st}}{\Gamma(t+1)} dt = \mu(e^{-s}, \beta) \quad \operatorname{Re} \beta > -1$$

$$\int_0^{\infty} \frac{t^{\beta} e^{-st}}{\Gamma(\alpha+t+1)} dt = e^{\alpha s} \mu(e^{-s}, \beta, \alpha) \quad \operatorname{Re} \beta > -1$$

which are equivalent to (1), (2) and show that the functions ν , μ are Laplace transforms of simple functions; and on the other hand to the formulas

$$(18) \int_0^{\infty} e^{-st} \nu(t) dt = (s \log s)^{-1} \quad \operatorname{Re} s > 1$$

$$\int_0^{\infty} e^{-st} \nu(t, \alpha) dt = s^{-\alpha-1} (\log s)^{-1} \quad \operatorname{Re} \alpha > -1, \quad \operatorname{Re} s > 1$$

$$(19) \int_0^{\infty} e^{-st} \mu(t, \beta) dt = s^{-1} (\log s)^{-\beta-1} \quad \operatorname{Re} s > 1$$

$$\int_0^{\infty} e^{-st} \mu(t, \beta, \alpha) dt = s^{-\alpha-1} (\log s)^{-\beta-1} \quad \operatorname{Re} \alpha > -1, \quad \operatorname{Re} s > 1$$

which may be established by means of (1), (2), (4) and show that ν and μ have very simple Laplace transforms. For derivations of many properties of the functions ν and μ by means of operational calculus, and for the application of these functions in operational calculus, see Barrucand and Colombo (1950), Colombo (1943, 1943 a, 1948), Humbert (1944, 1950),

Humbert and Poli (1944), Parodi (1945, 1947, 1948), and Poli (1946). Moreover, one of the numerous inversion formulas for the Laplace transformation

$$(20) f(s) = \int_0^{\infty} e^{-st} F(t) dt,$$

viz. the formula (Paley and Wiener 1934, p. 39, Doetsch 1937)

$$(21) F(t) = \lim_{\lambda \rightarrow \infty} \frac{1}{2\pi i} \int_0^{\infty} f(s) [\nu(st, -\frac{1}{2} + \lambda i) - \nu(st, -\frac{1}{2} - \lambda i)] ds$$

involves $\nu(x, a)$.

The integral formulas

$$(22) \int_0^{\infty} \exp\left(-\frac{x^2}{4y}\right) \mu(x, \beta, a) dx = 2^{\beta+1} y^{\frac{1}{2}} \pi^{\frac{1}{2}} \mu(y, \beta, \frac{1}{2}a)$$

$$\operatorname{Re} a > -1, \quad \operatorname{Re} y > 0$$

$$(23) \int_0^{\infty} x \exp\left(-\frac{x^2}{4y}\right) \mu(x, \beta, a) dx = 2^{\beta+2} \pi^{1/2} y^{3/2} \mu(y, \beta, \frac{1}{2}a - \frac{1}{2})$$

$$\operatorname{Re} a > -2, \quad \operatorname{Re} y > 0$$

$$(24) \int_0^{\infty} \exp\left(-\frac{x^2}{8y}\right) D_{\nu}\left(\frac{x}{2^{\frac{1}{2}} y^{\frac{1}{2}}}\right) \mu(x, \beta, a) dx$$

$$= 2^{\beta+\frac{1}{2}\nu+1} \pi^{\frac{1}{2}} y^{\frac{1}{2}\nu+\frac{1}{2}} \mu(y, \beta, \frac{1}{2}a - \frac{1}{2}\nu)$$

$$\operatorname{Re} a > -1, \quad \operatorname{Re} y > 0$$

may be established by substituting (4) in the integrands: in the last case, (24), use 8.3(20). These formulas show, in particular, that the functions ν, μ satisfy the following integral equations

$$(25) \frac{1}{2} \pi^{-\frac{1}{2}} y^{-\frac{1}{2}} \int_0^{\infty} \exp\left(-\frac{x^2}{4y}\right) \nu(x) dx = \nu(y)$$

$$\frac{1}{2} \pi^{-\frac{1}{2}} y^{-\frac{1}{2}} \int_0^{\infty} \exp\left(-\frac{x^2}{4y}\right) \mu(x, \beta) dx = 2^{\beta} \mu(y, \beta)$$

$$(26) \frac{1}{4} \pi^{-1/2} y^{-3/2} \int_0^{\infty} x \exp\left(-\frac{x^2}{4y}\right) \nu(x, -1) dx = \nu(y, -1)$$

$$\frac{1}{4} \pi^{-1/2} y^{-3/2} \int_0^{\infty} x \exp\left(-\frac{x^2}{4y}\right) \mu(x, \beta, -1) dx = 2^\beta \mu(y, \beta, -1)$$

$$(27) 2^{-\frac{1}{2}\nu-1} \pi^{-\frac{1}{2}} y^{-\frac{1}{2}\nu-\frac{1}{2}} \int_0^{\infty} \exp\left(-\frac{x^2}{8y}\right) D_{-a}\left(\frac{x}{2^{\frac{1}{2}} y^{\frac{1}{2}}}\right) \nu(x, a) dx$$

$$= \nu(y, a) \quad \text{Re } a > -1$$

$$2^{-\frac{1}{2}\nu-1} \pi^{-\frac{1}{2}} y^{-\frac{1}{2}\nu-\frac{1}{2}} \int_0^{\infty} \exp\left(-\frac{x^2}{8y}\right) D_{-a}\left(\frac{x}{2^{\frac{1}{2}} y^{\frac{1}{2}}}\right) \mu(x, \beta, a) dx$$

$$= 2^\beta \mu(y, \beta, a) \quad \text{Re } a > -1$$

In the case of the integral equation with the nucleus

$$\frac{1}{2\pi^{\frac{1}{2}} y^{\frac{1}{2}}} \exp\left(-\frac{x^2}{4y}\right)$$

it is known (Stanković 1953) that (25) gives all characteristic functions which, in a certain sense, are of regular growth; a similar statement is likely to be true in the case of (26) and (27). For other integral equations whose solutions involve the functions ν and μ see Colombo (1943 a, 1952) and Parodi (1948).

REFERENCES

- Agarwal, R.P., 1950: *Ann. Soc. Sci. Bruxelles. Ser. I.* 64, 164-168.
- Agarwal, R.P., 1951: *Bull. Calcutta Math. Soc.* 43, 153-167.
- Agarwal, R.P., 1953: *C.R. Acad. Sci. Paris* 236, 2031-2032.
- Agarwal, R.P., 1953 a: *Bull. Calcutta Math. Soc.* 45, 69-73.
- Aitken, A.C., 1939: *Determinants and matrices*, Oliver and Boyd, Edinburgh and London.
- Barnes, E.W., 1906: *Philos. Trans. Royal Soc. A* 206, 249-297.
- Barrucand, Pierre, 1951: *C.R. Acad. Sci. Paris* 232, 1058-1060.
- Barrucand, P.A. and Serge Colombo, 1950: *C.R. Acad. Sci. Paris* 230, 1335-1337.
- Bruwier, Laurent, 1949: *Bull. Soc. Roy. Sci. Liège* 18, 72-82.
- Bruwier, Laurent, 1949 a: *Bull. Soc. Roy. Sci. Liège* 18, 169-183.
- Bruwier, Laurent, 1949 b: *Mathesis* 58, 216-222.
- Buhl, Adolphe, 1925: *Séries analytique. Sommabilité. (Mem. Sci. Math. Fasc. 7)* Gauthier-Villars, Paris.
- Colombo, Serge, 1943: *Bull. Sci. Math. (2)* 67, 104-108.
- Colombo, Serge, 1943 a: *C.R. Acad. Sci. Paris* 216, 368-369.
- Colombo, Serge, 1948: *C.R. Acad. Sci. Paris* 226, 1235-1236.
- Colombo, Serge, 1950: *Ann. Télécomm.* 5, 347-364.
- Colombo, Serge 1952: *C.R. Acad. Sci. Paris* 235, 857-858, 928-929.
- Colombo, Serge, 1953: *Bull. Sci. Math. (2)* 77, 89-104.
- Copson, E.T., 1935: *An introduction to the theory of functions of a complex variable*. Oxford.
- Doetsch, Gustav, 1937: *Math. Z.* 42, 263-286.
- Ford, W.B., 1936: *The asymptotic developments of functions defined by Maclaurin series*. Univ. of Michigan.
- Grammel, Richard, 1948: *Arch. Math.* 1, 47-51.
- Grammel, Richard, 1948 a: *Ing.-Arch.* 16, 188-200.
- Grammel, Richard, 1950: *Ing.-Arch.* 18, 250-254.
- Hardy, G.H., 1940: *Ramanujan*. Cambridge.
- Humbert, Pierre, 1944: *C.R. Acad. Sci. Paris* 218, 99-100.

REFERENCES

- Humbert, Pierre, 1950: *C.R. Acad. Sci. Paris* 230, 504-505.
- Humbert, Pierre, 1953: *C.R. Acad. Sci. Paris* 236, 1467-1468.
- Humbert, Pierre and R.P. Agarwal, 1953: *Bull. Sci. Math.* (2) 77, 180-185.
- Humbert, Pierre and Paul Delerue, 1953: *C.R. Acad. Sci. Paris* 237, 1059-1060.
- Humbert, Pierre and Louis Poli, 1944: *Bull. Sci. Math.* (2) 68, 204-214.
- Lehrer, Yehiel, 1954: *Riveon Lematematika* 7, 71-73.
- Lindelöf, Ernst, 1903: *Bull. Sci. Math.* (2) 27, 213-226.
- Malmquist, Johannes, 1905: *Acta Math.* 29, 203-215.
- Mikusinski, Jan G., 1948: *Ann. Soc. Polon. Math.* 21, 46-51.
- Mittag-Leffler, G.M., 1903: *C.R. Acad. Sci. Paris* (2), 137, 554-558.
- Mittag-Leffler, G.M., 1904: *R. Accad. dei Lincei, Rendiconti* (5) 13, 3-5.
- Mittag-Leffler, G.M., 1905: *Acta Math.* 29, 101-182.
- Oniga, Théodore, 1948: *C.R. Acad. Sci. Paris* 227, 1138-1140.
- Paley, R.E.A.C. and Norbert Wiener, 1934: *Fourier transforms in the complex domain*. American Mathematical Society, New York.
- Parodi, Maurice, 1945: *Bull. Sci. Math.* (2) 69, 174-184.
- Parodi, Maurice, 1947: *C.R. Acad. Sci. Paris* 224, 780-782.
- Parodi, Maurice, 1947a: *Revue Sci.* 85, 360.
- Parodi, Maurice, 1948: *Ann. Soc. Sci. Bruxelles I*, 62, 24-26.
- Phragmén, Edvard, 1904: *Acta Math.* 28, 351-368.
- Poli, Louis, 1940: *Ann. Soc. Sci. Bruxelles I*, 60, 15-30.
- Poli, Louis, 1946: *C.R. Acad. Sci. Paris* 222, 580-581.
- Poli, Louis, 1949: *Ann. Univ. Lyon. Sect. A* (3) 12, 5-25.
- Poli, Louis, 1949a: *Cahiers Rhodaniens* 1, 1-15.
- Pollard, Harry, 1948: *Bull. Amer. Math. Soc.* 54, 1115-1116.
- Silverman, L.L., 1953: *Riveon Lematematika* 6, 53-60.
- Stanković, Bogoljub, 1953: *Rec. Acad. Serbe Sci.* 35, 95-106.
- Volterra, Vito, 1916: *Acc. dei Lincei, Memorie* (5) 11, 167-249
- Volterra, Vito and Joseph Peres, 1924: *Leçons sur la composition et les fonctions permutables*. Gauthier-Villars, Paris.

REFERENCES

- Wiman, Anders, 1905: *Acta Math.* 29, 191-201, 217-234.
- Wright, E.M., 1933: *J. London Math. Soc.* 8, 71-79.
- Wright, E.M., 1934: *Acta Math.* 63, 143-191.
- Wright, E.M., 1934 a: *Proc. London Math. Soc.* (2) 38, 257-270.
- Wright, E.M., 1940: *Quart. J. Math. Oxford Ser.* 11, 36-48.